Chapter 14
Quantum Nondemolition Measurements

Abstract Current attempts to detect gravitational radiation have to take into account the quantum uncertainties in the measurement process. Considering that the detectors are macroscopic objects in some cases as large as a 10-ton bar, the fact that quantum fluctuations in the detector must be taken into account seems surprising. However, as discussed in Chap. 8, gravitational waves interact so weakly with terrestrial detectors that a displacement of the order of 10^{-19} cm is expected. To illustrate how the measurement process may introduce uncertainties which obscure the signal we consider the simple example of a free mass. A measurement of the position of a free mass with a precision $\Delta x_i \approx 10^{-19}$ cm will disturb the momentum by an amount given by the uncertainty principles as $\Delta p \geq \hbar(2\Delta x_i)^{-1}$. The period of the gravitational waves is expected to be about 10^{-3} s, hence a second measurement of the position should be made after this time. During this period, however, the position uncertainty will grow under free evolution by an amount $\Delta x^2(\tau) = \Delta x^2(0) + [\Delta p^2(0)\tau^2/m^2]$. The following inequality then holds

$$\Delta x^2(\tau) \geq 2\Delta x(0)\Delta p(0)\frac{\tau}{m} \,. \tag{14.1}$$

Using the uncertainty principle we then find $\Delta x^2(\tau) \geq \hbar\tau/m$. Taking the detector mass equal to 10 tons, we find $\Delta x \geq 5 \times 10^{-19}$ cm. That is, the uncertainty introduced by the first measurement has made it impossible for a second measurement to determine with certainty whether a gravitational wave has acted or not. This is the standard quantum limit.

It is instructive to consider measurements of momentum instead of position. The first measurement of momentum causes an uncertainty in position. This however does not feed back to disturb the momentum as the momentum is a constant of motion for a free mass. Hence, subsequent determination of the momentum may be made with great predictability. The momentum of a free mass is an example of a quantum nondemolition (QND) variable. The concept of quantum nondemolition measurements has been introduced over the past few years to allow the detection, in principle, of very weak forces below the level of quantum noise in the detector. In the next section we will give a brief review of the concept of a quantum nondemolition measurement.

We mention here another way in which the standard quantum limit might be overcome. Quantum nondemolition measurements generally presume that nothing at all is known about the state of the system to be measured. The standard quantum limit for a free mass, for example, was derived by assuming no correlation between position and momentum. If however we are permitted to prepare the state of the system to be measured, the accuracy of a measurement can be improved without resort to a QND scheme. For example, in the case of a free particle the position variance at time τ is given by

$$\Delta x^2(\tau) = \Delta x^2(0) + \frac{\Delta p^2(0)\tau^2}{m^2} + \langle \Delta x(0)\Delta p(0) + \Delta p(0)\Delta x(0)\rangle \frac{\tau}{m} \tag{14.2}$$

where the possibility of nonzero correlation between position and momentum has been included. In fact, this correlation may be negative if the initial state of the particle is chosen to be a 'contractive state'. If this is the case it is clear that at a later time τ it is possible that $\Delta x^2(\tau) < \hbar\tau/m$, thus allowing a greater accuracy than the standard quantum limit.

14.1 Concept of a QND Measurement

The basic requirement of a QND measurement is the availability of a variable which may be measured repeatedly giving predictable results in the absence of a gravitational wave [1]. Clearly this requires that the act of measurement itself does not degrade the predictability of subsequent measurements. Then in a sufficiently long sequence of measurements the output becomes predictable.

This requirement is satisfied if for an observable $A^I(t)$ (in the interaction picture)

$$[A^I(t),\, A^I(t')] = 0\,. \tag{14.3}$$

The condition ensures that if the system is in an eigenstate of $A^I(t_0)$ it remains in this eigenstate for all subsequent times although the eigenvalues may change. Such observables are called *QND observables*. Clearly constants of motion will be QND observables. Thus for a free particle, energy and momentum are QND observables while the position is not as

$$x(t+\tau) = x(t) + p\frac{\tau}{m} \tag{14.4}$$

and

$$[x(t),\, x(t+\tau)] = \frac{\mathrm{i}\hbar\tau}{m}\,. \tag{14.5}$$

For a harmonic oscillator of unit mass

$$[x(t),\, x(t+\tau)] = \frac{\mathrm{i}\hbar}{\omega}\sin\,\omega\tau\,. \tag{14.6}$$

and

$$[p(t),\ p(t+\tau)] = \mathrm{i}\hbar\omega \ \sin\ \omega\tau\ , \tag{14.7}$$

thus position and momentum are not QND observables for the harmonic oscillator.

There are, however, QND observables for the harmonic oscillator. We define the explicitly time dependent quadrature phase amplitudes for the oscillator as follows.

$$X_1(t) = a\mathrm{e}^{\mathrm{i}\omega t} + a^\dagger \mathrm{e}^{-\mathrm{i}\omega t} \tag{14.8}$$

and

$$X_2(t) = -\mathrm{i}(a\mathrm{e}^{\mathrm{i}\omega t} - a^\dagger \mathrm{e}^{-\mathrm{i}\omega t})\ . \tag{14.9}$$

In the Heisenberg picture the quadrature phase operators are given by

$$X_1 = a + a^\dagger\ , \tag{14.10}$$

$$X_2 = -\mathrm{i}(a - a^\dagger)\ , \tag{14.11}$$

which clearly shows that the quadrature phase operators are constants of the motion. In terms of the position and momentum the quadrature phase operators are

$$X_1(t) = \left(\frac{2\omega}{\hbar}\right)^{1/2} [x(t)\cos\ \omega t - \frac{p(t)}{\omega} \sin\ \omega t] \tag{14.12}$$

and

$$X_2(t) = \left(\frac{2\omega}{\hbar}\right)^{1/2} [x(t)\sin\ \omega t + \frac{p(t)}{\omega} \cos\ \omega t]\ . \tag{14.13}$$

Thus the X_1 and X_2 axes rotate with respect to the position and momentum axes of phase space, at frequency ω.

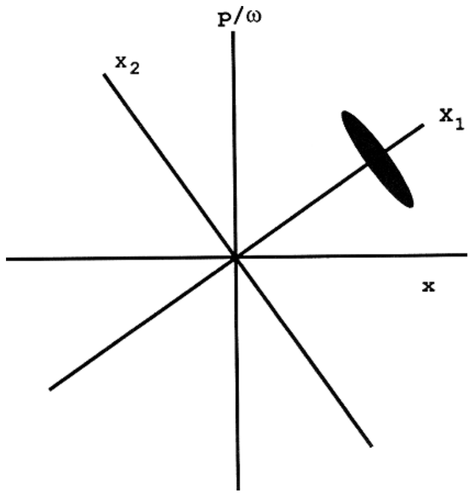

Fig. 14.1 Error box in the phase plane for a harmonic oscillator. The error box rotates with respect to the x and p/ω axes but is stationary with to the X_1, X_2 axes

The behaviour of X_1 and X_2 are most easily discussed with reference to an amplitude and phase diagram. In such a diagram the state of the system is represented by a set of points centred on the mean and contained within an error ellipse determined by the variance of the quadrature phases. Alternatively the error box may be regarded as a contour of the Wigner function. In Fig. 14.1 an error ellipse for the oscillator is shown. The error ellipse is stationary with respect to the X_1 and X_2 axes but rotates with respect to the x and p axes. This clearly illustrates how uncertainties in momentum feed back into position.

14.2 Back Action Evasion

Having first determined the QND variables of the detector it is necessary to couple the detector to a readout system or meter. It is essential that the coupling to the meter does not feed back fluctuations into the QND variable of the detector. In order to avoid this it is sufficient if the QND variable A commutes with the Hamiltonian coupling the detector and the meter, H_{DM}, that is

$$[A, H_{\mathrm{DM}}] = 0 \tag{14.14}$$

This is known as the *back action evasion criterion*.

In this chapter we are primarily concerned with QND measurements on optical systems. This requires a slight change in nomenclature. We will refer to the field with respect to which the QND variable is defined as the 'signal' rather than the detector, and the field upon which measurements are ultimately made as the 'probe' rather than the meter.

14.3 Criteria for a QND Measurement

We need to clearly define the objectives of a quantum nondemolition measurement in an optical context. These objectives may differ depending on the situation of the measurement. For example, in a transmission with a series of receivers, the goal may be to tap information from the signal, without degrading the signal transmitted to the next receiver. In a system used to measure the magnitude of an external force the goal of the measurement may be state preparation. That is an initial measurement prepares the system in a known quantum state. The presence of the perturbing force will be detected by a subsequent measurement on the system. In order to evaluate the merits of a measurement scheme we shall define a set of criteria which we would like to be satisfied in a good measurement. We begin by considering the general measurement scheme depicted in Fig. 14.2 where an observable X_{in} of the input signal is determined by a measurement of an observable Y_{out} of the output probe. The measurement may be characterised by the following criteria [2]:

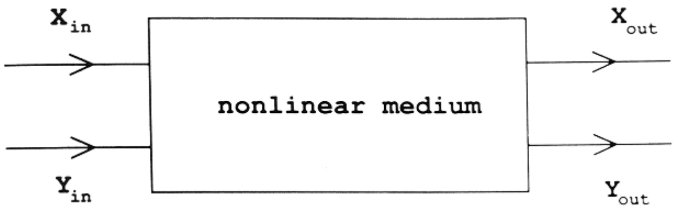

Fig. 14.2 General scheme for a QND measurement in an optical context

1. *How good is the measurement scheme?* This is determined by the level of correlation between the probe field measured by a detector and the signal field incident on the apparatus. The appropriate correlation function is

$$C^2_{X^{\text{in}}Y^{\text{out}}} = \frac{|\langle X^{\text{in}}\, Y^{\text{out}}\rangle_{\text{s}} - \langle X^{\text{in}}\rangle\langle Y^{\text{out}}\rangle|^2}{V_{X^{\text{in}}}V_{Y^{\text{out}}}} \tag{14.15}$$

where $V(A) = \langle A^2\rangle - \langle A\rangle^2$ is the variance in a measurement of A and $\langle AB\rangle_{\text{s}} = \langle AB + \text{BA}\rangle/2$. For a perfect measurement device the phase quadrature of the probe output is equal to the amplitude quadrature of the signal input multiplied by the QND gain, plus the input probe phase quadrature. In this case the correlation coefficient defined above is unity, for large gain.

2. *How much does the scheme degrade the signal?* The quantity of interest here is the correlation between the signal input field and the signal output field:

$$C^2_{X^{\text{in}}X^{\text{out}}} = \frac{|\langle X^{\text{in}}\, X^{\text{out}}\rangle_{\text{s}} - \langle X^{\text{in}}\rangle\langle X^{\text{out}}\rangle|^2}{V_{X^{\text{in}}}V_{X^{\text{out}}}} \tag{14.16}$$

This is a measure of the back action evasion, that is the ability of the scheme to isolate quantum noise introduced by the measurement process from the observable of interest. For an ideal QND scheme we require this correlation to be unity. Thus, for a perfect QND scheme we have $C^2_{X^{\text{in}}Y^{\text{out}}} + C^2_{X^{\text{in}}X^{\text{out}}} = 2$.

3. *How good is the scheme as a state preparation device?* If we have a perfect measurement device that does not degrade the signal at all, we satisfy the two previous criteria exactly, then we must be able to completely predict the state of the signal output. However, once we leave this ideal case the predictability of the signal output is no longer fully determined by correlations with the signal input. The extreme example is that of a destructive measurement: independently of how well the input is measured the output is always the vacuum. On the other hand, the correlation between the signal and probe output fields is not a good indicator of the quality of state preparation. Figure 14.3 shows a situation in which both output fields are well correlated, but a probe measurement does not allow inference of the signal output field to be better than the quantum limit. This situation arises when the interaction within the QND medium introduces significant correlated noise to both output fields.

Given that we have made a perfect measurement of some physical quantity X with the result x, what is the state of the system after such a measurement

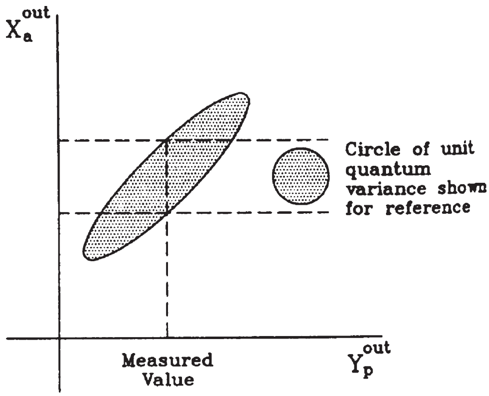

Fig. 14.3 Illustration of a situation in which a value of the probe output has been measured, but when mapped onto the error ellipse, does not permit an inference of the signal to better than the quantum limit

conditioned on the result x? In standard quantum mechanics the conditional state is generally assumed to be an eigenstate of X with an eigenvalue equal to the measured result, at least for perfect measurements. In the case of the QND measurement scheme above we then expect the state of the signal mode conditioned on the results of the probe measurements should in some limit be an eigenstate of X^{out}. Of course, the variance of X^{out} in such a state is zero. Thus as a measure of how well the scheme prepares eigenstates at the output we need to consider the conditional variance $V(X^{\text{out}}|Y^{\text{out}})$. This quantity is calculated as follows:

The probability to obtain the result Y^{out} for a probe measurement is given by

$$P(Y^{\text{out}}) = \text{Tr}\{\rho^{\text{out}}|Y^{\text{out}}\rangle\langle Y^{\text{out}}|\} \tag{14.17}$$

(assuming perfect readout of the probe state). The conditional state of the signal mode based on this result is

$$\rho^{\text{out}} = \frac{\text{Tr}_{\text{probe}}\{\rho^{\text{out}}|Y^{\text{out}}\rangle\langle Y^{\text{out}}|\}}{P(Y^{\text{out}})} \,. \tag{14.18}$$

Using this result we see that the conditional distribution for X^{out} is

$$\begin{aligned} P(X^{\text{out}}|Y^{\text{out}}) &= \text{Tr}_{\text{signal}}\{\rho^{\text{out}}_{\text{signal}}|X^{\text{out}}\rangle\langle X^{\text{out}}|\} \\ &= \frac{P(X^{\text{out}},\, Y^{\text{out}})}{P(Y^{\text{out}})} \end{aligned}$$

where

$$P(X^{\text{out}},\, Y^{\text{out}}) = \text{Tr}\{\rho^{\text{out}}|X^{\text{out}}\rangle\langle X^{\text{out}}| \otimes |Y^{\text{out}}\rangle\langle Y^{\text{out}}|\} \,. \tag{14.19}$$

In many cases of interest $P(X^{\text{out}},\, Y^{\text{out}})$ is a bivariate Gaussian. In that case one may show

$$V(X^{\text{out}}|Y^{\text{out}}) = V(X^{\text{out}})(1 - C^2_{X^{\text{out}}Y^{\text{out}}}) \,. \tag{14.20}$$

Thus, the condition for a perfect state reduction in the conditional state is

$$C^2_{X^{\mathrm{out}}Y^{\mathrm{out}}} = 1 \ . \tag{14.21}$$

We shall now analyse some possible measurement schemes and see how well they approach the conditions for an ideal measurement.

14.4 The Beam Splitter

We consider first a beam splitter deflecting part of the incident signal field onto a homodyne detector, as shown in Fig. 14.4. This will serve as a standard of comparison for other measurement schemes. There is obviously little point in constructing complicated schemes involving cavities containing nonlinear media if they cannot improve on the performance of a beam splitter. We consider the case where the signal and probe fields are single mode with annihilation operators a and b, respectively. The amplitude and phase quadratures of the signal and probe fields are defined as

$$X_{\mathrm{a}} = a + a^{\dagger} \ , \tag{14.22}$$

$$X_{\phi} = -\mathrm{i}(a - a^{\dagger}) \ , \tag{14.23}$$

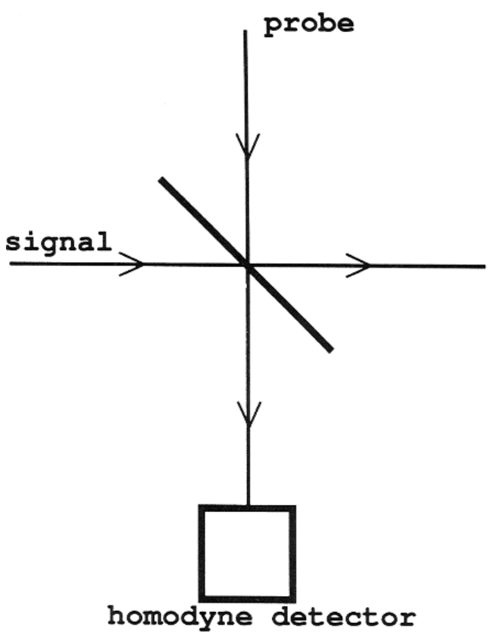

Fig. 14.4 An optical measurement scheme for the quadrature phase based on a beam splitter

$$Y_{\mathrm{a}} = b + b^{\dagger} , \tag{14.24}$$

$$Y_{\phi} = -\mathrm{i}(b + b^{\dagger}) . \tag{14.25}$$

(This notation assumes that the coherent amplitude of each field is real). Note that according to the uncertainty principle

$$\Delta X_{\mathrm{a}} \Delta X_{\phi} \geq 1 \tag{14.26}$$

where ΔX_{a} and ΔX_{ϕ} are the square root of the variances. A precise measurement of the amplitude quadrature must therefore be at the expense of uncertainty in the phase. A good back action evading scheme must be able to feed all the quantum noise induced by the act of measurement into the phase quadrature of the signal.

In the beam splitter the phase change on reflection gives a coupling between the amplitude quadrature of the signal and the phase quadrature of the probe. We consider making a measurement on the phase quadrature of the probe in order to determine the amplitude quadrature of the signal. The input quadrature fields can be related to the output quadrature fields using the transformation at the beam splitter:

$$\begin{pmatrix} X_{\mathrm{a}}^{\mathrm{out}} \\ Y_{\phi}^{\mathrm{out}} \end{pmatrix} = \begin{pmatrix} \sqrt{1-\eta^2} & -\eta \\ \eta & \sqrt{1-\eta^2} \end{pmatrix} \begin{pmatrix} X_{\mathrm{a}}^{\mathrm{in}} \\ Y_{\phi}^{\mathrm{in}} \end{pmatrix} \tag{14.27}$$

where η is real and represents the mirror amplitude reflectivity, and there is a $\pi/2$ phase change upon reflection.

The first criterion for a good QND measurement scheme is that it must be a good back action evading device. In other words, it must be able to isolate the signal field from quantum noise introduced by the measurement. How well the beam splitter achieves this is represented by the correlation between the input and the output signal fields,

$$C^2_{X_{\mathrm{a}}^{\mathrm{in}} X_{\mathrm{a}}^{\mathrm{out}}} = \frac{(1-\eta^2) V_{X_{\mathrm{a}}^{\mathrm{in}}}}{(1-\eta^2) V_{X_{\mathrm{a}}^{\mathrm{in}}} + \eta^2 V_{Y_{\phi}^{\mathrm{in}}}} \tag{14.28}$$

where $V_{X_{\mathrm{a}}^{\mathrm{in}}}$ denotes the variance of the signal input, and $V_{Y_{\phi}^{\mathrm{in}}}$ is the corresponding variance for the probe. These quantities are a measure of the quantum or classical noise present in the input fields at the appropriate qudrature phase. For a beam splitter with 50% reflectivity, the correlation between the signal input and output is given by the ratio of the signal noise to the total noise introduced to the system through both input ports.

The second criterion reflects how well the scheme acts as a measurement device. The readout measurement is made on the probe output field, so the level of correlation between this quantity and the signal field incident on the device determines how well a measurement can be made. The appropriate correlation function is

$$C^2_{X_{\mathrm{a}}^{\mathrm{in}} Y_{\phi}^{\mathrm{out}}} = \frac{\eta^2 \, V_{X_{\mathrm{a}}^{\mathrm{in}}}}{\eta^2 \, V_{X_{\mathrm{a}}^{\mathrm{in}}} + (1-\eta^2) V_{Y_{\phi}^{\mathrm{in}}}} . \tag{14.29}$$

Again, for a 50% beam splitter, the correlation is given by the ratio of the incident signal noise to the total noise introduced.

The third criteria is that the measurement must prepare the output observable in a well known state. This is given by the variance in the output state after the measurement has been performed. Using

$$C^2_{X_a^{\rm out}Y_\phi^{\rm out}} = \frac{\eta^2(1-\eta^2)(V_{X_a^{\rm in}} - V_{Y_\phi^{\rm in}})^2}{[(1-\eta^2)V_{X_a^{\rm in}} + \eta^2 V_{Y_\phi^{\rm in}}][\eta^2 V_{X_a^{\rm in}} + (1-\eta^2)V_{Y_\phi^{\rm in}}]} \tag{14.30}$$

and

$$V_{X_a^{\rm out}} = (1-\eta^2)V_{X_a^{\rm in}} + \eta^2\, V_{Y_\phi^{\rm in}}\ , \tag{14.31}$$

and the linearity predictor for the beam splitter, the conditional variance is given by

$$V(X_a^{\rm out}|Y_\phi^{\rm out}) = \frac{V_{X_a^{\rm in}}\, V_{Y_\phi^{\rm in}}}{\eta^2\, V_{X_a^{\rm in}} + (1-\eta^2)V_{Y_\phi^{\rm in}}}\ . \tag{14.32}$$

We would like this variance to be zero. If both signal and probe inputs are in the vacuum or coherent states with unit quantum variance in both quadratures, then

$$\begin{aligned}
&C^2_{X_a^{\rm in}Y_\phi^{\rm out}} = \eta^2,\\
&C^2_{X_a^{\rm in}X_a^{\rm out}} = 1-\eta^2,\\
&C^2_{X_a^{\rm out}Y_\phi^{\rm out}} = 0\\
&V(X_a^{\rm out}|Y_\phi^{\rm out}) = 1.
\end{aligned}$$

As expected, the correlation between the signal input field and the signal output field is the intensity transmission coefficient of the mirror. To reduce the amount of noise added to the signal variable we would like to split off only a small portion of the light field. However, this reduces the correlation between the signal input field and the probe field upon which the readout is made, which is given by the intensity reflection coefficient. It is not possible therefore to simultaneously satisfy the first two criteria for a good QND scheme. Since the signal and probe fields are completely uncorrelated, a measurement of the probe does not reduce the signal output variable at all. The result is that you cannot use a beam splitter to prepare the state of the output signal with probe fluctuations at the vacuum level. Clearly the performance of the beam splitter improves if the input probe has squeezed fluctuations (Exercise 14.5).

Note that for the beam splitter $C^2_{X_a^{\rm in}Y_\phi^{\rm out}} + C^2_{X_a^{\rm in}Y_a^{\rm out}} = 1$, a typical result for a non-back-action evasion scheme, and significantly less than the maximum result of 2 for this quantity achieved in an ideal scheme. The quality of a QND scheme can thus be measured by the extent to which this quantity exceeds unity and approaches the upper limit of 2.

14.5 Ideal Quadrature QND Measurements

We now consider another scheme to make perfect QND measurements of the quadrature phase of a single mode field. We shall assume that the amplitude quadratures of each mode are coupled. That is, the interaction Hamiltonian has the form

$$\mathscr{H} = \hbar\chi\, X_{\mathrm{a}}\, Y_{\mathrm{a}} \tag{14.33}$$

where χ is the coupling strength and X_{a}, Y_{a} are defined by (14.22 and 14.24). Clearly X_{a} is a QND variable of the signal which satisfies the back action evading condition (14.12). The input and output quadratures are related by

$$\begin{aligned} X_{\mathrm{a}}^{\mathrm{out}} &= X_{\mathrm{a}}^{\mathrm{in}}\,, \\ Y_{\phi}^{\mathrm{out}} &= G\, X_{\mathrm{a}}^{\mathrm{in}} + Y_{\phi}^{\mathrm{in}}\,, \end{aligned}$$

where $G = \chi t$ is known as the QND gain (t being the interaction time).

A measurement on the phase quadrature of the probe will be used to determine the amplitude quadrature of the input signal. To begin we calculate the correlation coefficients which define the measurement. Clearly $C_{X_{\mathrm{a}}^{\mathrm{in}} X_{\mathrm{a}}^{\mathrm{out}}} = 1$, the signal is completely unaffected by the measurement. The correlation between the input signal and the phase of the output probe is

$$C_{X_{\mathrm{a}}^{\mathrm{in}} Y_{\phi}^{\mathrm{out}}} = \frac{G^2 V_{X_{\mathrm{a}}^{\mathrm{in}}}}{G^2 V_{X_{\mathrm{a}}^{\mathrm{in}}} + V_{Y_{\phi}^{\mathrm{in}}}} \tag{14.34}$$

where we have taken $\langle Y_{\phi}^{\mathrm{in}} \rangle = 0$. For a large QND gain $G^2 \gg 1$,

$$C_{X_{\mathrm{a}}^{\mathrm{in}} Y_{\phi}^{\mathrm{out}}} \to 1\,. \tag{14.35}$$

The conditional variance $V(X_{\mathrm{a}}^{\mathrm{out}} | Y_{\phi}^{\mathrm{out}})$ which determines the value of the scheme as a state preparation device is given by

$$\begin{aligned} V(X_{\mathrm{a}}^{\mathrm{out}} | Y_{\phi}^{\mathrm{out}}) &= V_{X_{\mathrm{a}}^{\mathrm{in}}} \left(1 - \frac{G^2 V_{X_{\mathrm{a}}^{\mathrm{in}}}}{G^2 V_{X_{\mathrm{a}}^{\mathrm{in}}} + V_{Y_{\phi}^{\mathrm{in}}}} \right) \\ &\approx \frac{V_{Y_{\phi}^{\mathrm{in}}}}{G^2} \\ &\to 0 \text{ for } G^2 \gg 1. \end{aligned}$$

Again in the limit of high QND gain this device operates as a good state preparation device.

Another measure of the performance of the measurement is the signal-to-noise ratio of the probe output

$$\frac{\text{signal}}{\text{noise}} = \frac{\langle Y_\phi^{\text{out}} \rangle^2}{V_{Y_\phi^{\text{out}}}}$$
$$= \frac{G^2 \langle X_{\text{a}}^{\text{in}} \rangle^2}{G^2 V_{X_{\text{a}}^{\text{in}}} + V_{Y_\phi^{\text{in}}}}$$
$$\to \frac{\langle X_{\text{a}}^{\text{in}} \rangle^2}{V_{X_{\text{a}}^{\text{in}}}} \text{ for } G^2 \gg 1.$$

In the limit of large QND gain the signal-to-noise ratio of the output probe is equal to that of the signal input.

14.6 Experimental Realisation

It is possible to achieve a QND coupling of the form in (14.33) by considering two degenerate modes a and b with frequency ω and orthogonal polarisation, which undergo parametric amplification [3]. The two polarisation modes initially undergo a mixing interaction using polarisation rotators, after which a mixture of the signal and probe fields will propagate along each of the ordinary and orthogonal extrordinary axis of a KTP crystal pumped by a pulsed intense classical field. After this amplification step the fields then pass through a second polarisation rotator adjusted to give the same mixing angle as the first. In order to ensure that the device operates as an ideal QND scheme the mixing angle of the rotators must be carefully adjusted. The situation is depicted in Fig. 14.5.

The transformation performed by the polarisation rotators is given by

$$a(\theta) = a\cos\theta + \mathrm{i}b\sin\theta\ , \tag{14.36}$$

$$b(\theta) = b\cos\theta + \mathrm{i}a\sin\theta\ , \tag{14.37}$$

with θ being the mixing angle. The transformation in the parametric amplification process is

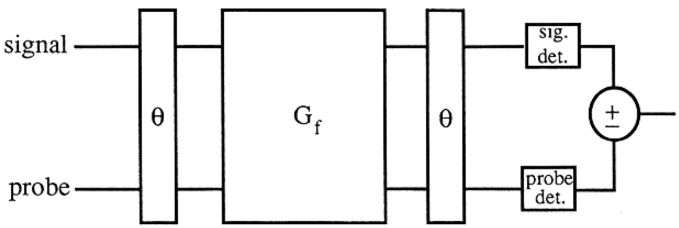

Fig. 14.5 Schematic representation of a perfect QND scheme based on a parametric interaction

$$a(r) = a \cosh r + \mathrm{i} b^\dagger \sinh r \,, \tag{14.38}$$

$$b(r) = b \cosh r + \mathrm{i} a^\dagger \sinh r \,, \tag{14.39}$$

where r is related to the parametric gain G_f by $G_\mathrm{f} = \mathrm{e}^r$. Using these transformations for the system in Fig. 14.5, we find that the transformations for the signal and probe quadratures are

$$X_\mathrm{a}^\mathrm{out} = X_\mathrm{a}^\mathrm{in} \,, \tag{14.40}$$

$$X_\phi^\mathrm{out} = X_\phi^\mathrm{in} + (G_\mathrm{f} - G_\mathrm{f}^{-1}) Y_\phi^\mathrm{in} \,, \tag{14.41}$$

$$Y_\mathrm{a}^\mathrm{out} = Y_\mathrm{a}^\mathrm{in} - (G_\mathrm{f} - G_\mathrm{f}^{-1}) X_\mathrm{a}^\mathrm{in} \,, \tag{14.42}$$

$$Y_\phi^\mathrm{out} = Y_\phi^\mathrm{in} \,, \tag{14.43}$$

where we have taken the polarisation mixing angle to be

$$\theta = \mathrm{arc\ cos} \left(\frac{G_\mathrm{f} + 1}{\sqrt{2(G_\mathrm{f}^2 + 1)}} \right) . \tag{14.44}$$

Clearly this represents an ideal QND scheme.

In the experiment of *La Porta* et al. [3], the incident signal was in a coherent state while the probe was in the vacuum state at input. The measured gain was $G_\mathrm{f} = 1.33$ thus giving $\theta = 8^\circ$. The output quadratures are measured by phase sensitive homodyne detection using polarisation beam splitters. To demonstrate that the experiment is operating as a back action evading measurement, three quantities were measured. Firstly the variances of each quadrature of the signal and probe were measured, with the gain both on and off. The probe shows a large noise in only one quadrature when the paramp was on. This is a reflection of the gain term appearing in the amplitude quadrature of the probe. The phase quadrature noise was at the shot-noise level. Secondly, the signal variances alone were measured showing a similar effect. Finally the variance of the quantities

$$X_\pm = (G_\mathrm{f} - G_\mathrm{f}^{-1}) X_\mathrm{a}^\mathrm{out} \pm Y_\mathrm{a}^\mathrm{out} \tag{14.45}$$

was measured. For the choosen input states one easily verifies that

$$V(X_+) = 1 \,, \tag{14.46}$$

$$V(X_-) = 4(G_\mathrm{f} - G_\mathrm{f}^{-1})^2 \,. \tag{14.47}$$

These quantities were measured by adjusting the relative gain of the photo-current amplifiers to weight the $X_\mathrm{a}^\mathrm{out}$ quadrature as indicated. This ensures that any correlation between $X_\mathrm{a}^\mathrm{out}$ and $Y_\mathrm{a}^\mathrm{out}$ will give maximum cancellation of the noise from each quadrature seperately in the variances for $X_\pm$. The results of this experiment are

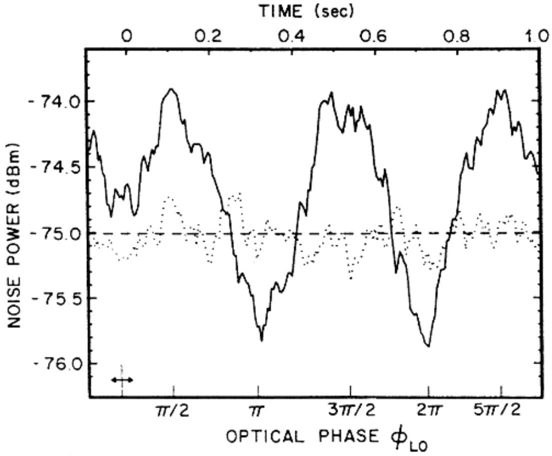

Fig. 14.6 The combination of the amplified signal and probe quadrature at output versus the phase of the local oscillator. The variance of X – occurs at integer multiples of π. The *dotted line* corresponds to the case of the paramp off, and is identical to randomly added shot-noise levels of the signal and probe. The noise reduction is 0.6 dB below the combined shot-noise level. From Arthur La Porta: Phys. Rev. Lett. **62**, 28 (1989)

shown in Fig. 14.6. The results of all experiments taken together clearly indicate that the scheme is operating as a back action evasion device.

Recently, QND experiments have been performed [4] with two-photon transitions in three-level atoms, where the signal amplitude is strongly correlated with the probe phase. The measurement correlation between the signal in and the probe out is $C^2_{X_a^{\text{in}}\, Y_\phi^{\text{out}}} = 0.45$ and the back action evasion correlation between signal in and signal out is $C^2_{X_a^{\text{in}}\, Y_\phi^{\text{out}}} = 0.9$. The overall performance measure of this device as a QND optical tap is then determined by $C^2_{X_a^{\text{in}}\, Y_\phi^{\text{out}}} + C^2_{X_a^{\text{in}}\, X_\phi^{\text{out}}} = 1.35$, which exceeds the beam splitter limit of one, but is still well below the optimal value of 2.

14.7 A Photon Number QND Scheme

We turn now to a scheme to measure the photon number in the signal field. Conventional photon counting techniques absorb quanta. The scheme considered is a true nondemolition measurement of photons in that no photons are absorbed from the signal field [5].

Consider the coupled signal/probe system described by the interaction Hamiltonian

$$\mathscr{H}_1 = \hbar\chi a^\dagger a b^\dagger b \tag{14.48}$$

where a refers to the signal mode, and b to the probe. Such a coupling can occur in a four wave mixing process in which case χ is proportional to the third-order nonlinear susceptibility.

Clearly, $a^\dagger a$ is a constant of the motion and is thus a QND variable for the signal. The solution of the Heisenberg equations of motion gives

$$a(t) = \mathrm{e}^{-\mathrm{i}\chi b^\dagger b t} a(0) \,, \tag{14.49}$$

$$b(t) = \mathrm{e}^{-\mathrm{i}\chi a^\dagger a t} b(0) \,. \tag{14.50}$$

These equations describe a mutual intensity-dependent phase shift for the signal and probe fields. If we can measure this phase shift on the probe, information on the signal photon number may be obtained. The probe phase shift may, in fact, be determined by homodyne detection of a probe quadrature.

Using (14.24 and 14.25) the phase quadrature for the probe field becomes

$$Y_\phi^{\mathrm{out}} = \cos(\kappa a^\dagger a)\, Y_\phi^{\mathrm{in}} - \sin(\kappa a^\dagger a)\, Y_{\mathrm{a}}^{\mathrm{in}} \tag{14.51}$$

where $\kappa = \chi t$. It would appear from this equation that the signal operator that we actually measure is not simply $a^\dagger a$ but a nonlinear function of $a^\dagger a$. However in any practical scheme κ is so small that we may approximate the trignometric functions by the lowest order in κ. Thus, we use

$$Y_\phi^{\mathrm{out}} = Y_\phi^{\mathrm{in}} - \kappa a^\dagger a\, Y_{\mathrm{a}}^{\mathrm{in}} \,. \tag{14.52}$$

What quantity plays the role of the QND gain in this scheme? To answer this question we need to evaluate the correlation functions which provide criteria for the quality of the QND measurement. The first of these functions is

$$\begin{aligned} C^2_{a^\dagger a Y_\phi^{\mathrm{out}}} &= \frac{|\langle a^\dagger a Y_\phi^{\mathrm{out}}\rangle - \langle a^\dagger a\rangle\langle Y_\phi^{\mathrm{out}}\rangle|^2}{V(a^\dagger a)V(Y_\phi^{\mathrm{out}})} \\ &= \frac{\kappa^2\langle Y_{\mathrm{a}}^{\mathrm{in}}\rangle^2 V(a^\dagger a)}{V(Y_\phi^{\mathrm{in}}) + 2\kappa F_1 + \kappa^2 F_2} \end{aligned}$$

where

$$\begin{aligned} F_1 &= \langle a^\dagger a\rangle\langle Y_{\mathrm{a}}^{\mathrm{in}},\, Y_\phi^{\mathrm{in}}\rangle_{\mathrm{s}} \,, \\ F_2 &= V(a^\dagger a)\langle Y_\phi^{\mathrm{in}}\rangle^2 + V(Y_{\mathrm{a}}^{\mathrm{in}})(V(a^\dagger a) + \langle a^\dagger a\rangle^2) \end{aligned}$$

and the symmetrised correlation function is defined by

$$\langle A, B\rangle_{\mathrm{s}} = \frac{1}{2}\langle AB + BA\rangle - \langle A\rangle\langle B\rangle \,. \tag{14.53}$$

If we now assume $\langle Y_{\mathrm{a}}^{\mathrm{in}}\rangle^2 \gg V(Y_{\mathrm{a}}^{\mathrm{in}}),\ V(Y_\phi^{\mathrm{in}})$ (that is the coherent amplitude of the probe is much greater than the fluctuations in either quadrature), we find,

$$C^2_{a^\dagger a Y_\phi^{\mathrm{out}}} \to 1 \tag{14.54}$$

when $\langle Y_{\mathrm{a}}^{\mathrm{in}}\rangle$ is large. It would thus appear that the coherent amplitude of the probe plays the role of the QND gain. This result is easily understood in terms of a complex amplitude diagram for the probe. If the vector representing the input state of the probe is very long a small rotation due to the signal makes a large change in the projection of the coherence vector onto the phase quadrature direction. In a similar way the signal-to-noise ratio of the output quadrature reduces to the signal-to-noise ratio for $a^\dagger a$ in the limit of $\langle Y_\phi^{\mathrm{in}}\rangle \gg 1$. One easily verifies that the conditional variance of $a^\dagger a$ at the output approaches zero in the same limit. This last result indicates that the conditional state of the signal output will have sub-Poisonnian statistics.

If κ is not small, we cannot simply approximate the coupling between the signal and probe as being linear in the signal photon number. A measurement of the probe quadrature phase still provides information on the signal photon number, however, due to the multivalued nature of the trignometric functions, the signal is reduced to a superposition of number states in the case that the initial photon number distribution of the signal is sufficiently broad [5].

Exercises

14.1 Consider a signal beam and a probe beam coupled via a four wave mixing interaction;

$$\mathscr{H}_{\mathrm{I}} = \hbar\chi a^\dagger a b^\dagger b \tag{14.55}$$

Calculate the QND correlation coefficients between the amplitude quadrature of the signal $X_{\mathrm{a}} = a^\dagger + a$ and the phase quadrature of the probe, $Y_\phi = -\mathrm{i}(b - b^\dagger)$.

14.2 Consider a QND measurement in an optical cavity. Generalise the QND correlations to the frequency domain. For example, the stationary spectral covariance between signal X and probe Y is defined as

$$C_{XY}(\omega) = \int \mathrm{d}\tau \mathrm{e}^{-i\omega\tau}\frac{1}{2}\langle X(t)\, Y(t+\tau) + Y(t+\tau)X(t)\rangle \tag{14.56}$$

Using the input/output formulation developed in Chap. 7, calculate the spectral QND correlations between the signal amplitude and the probe phase in the input and output fields for the intracavity interaction

$$\mathscr{H}_{\mathrm{I}} = \hbar\frac{\chi}{2}X_{\mathrm{a}}Y_\phi\ . \tag{14.57}$$

14.3 Consider the four wave mixing process with the Hamiltonian

$$\mathscr{H}_{\mathrm{I}} = \hbar\chi a^\dagger a b^\dagger b \tag{14.58}$$

taking place inside a cavity. Calculate the QND spectral correlations between the amplitude quadrature X_{a} of the signal and the phase quadrature Y_ϕ of the probe, in the input and output fields.

14.4 Consider a degenerate parametric amplifier inside a resonant two-sided optical cavity with mirrors with loss rates γ_1, γ_2. Treat the left-hand input as the signal and the right-hand input as the probe. Calculate the QND spectral correlations between the phase quadratures of the signal and the probe.

14.5 Calculate the QND correlations for a beam splitter with a squeezed input probe.

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